Are Limit Orders Rational?[#]

Martin Šmíd[*]

According to textbook economics, prices are determined by intersections of demand and supply curves. In practice, however, there is no arbiter investigating the curves and determining equilibriums; instead, some agents offer prices and some others react to them by accepting them or not. With a few exceptions (e.g. auctions) all markets follow the offer-reaction pattern, including today's financial markets which we deal with in the present paper.

In financial markets, the offers mentioned above are called *buy* or *sell limit orders* while the reactions (acceptations) are known as *buy* or *sell market orders*. Each buy (sell) limit order comprises a *limit price,* i.e. the highest (lowest) price the agent is content to buy (sell) for, and the *order size,* i.e. the demanded (offered) amount of the traded commodity. Market orders consist of an order size only; once a buy (sell) market order is submitted, the trade is made for the most favorable price among the waiting sell (buy) orders (when the size of the best limit order is too small to satisfy the market order, the second best limit order is used etc.[1]). A more detailed description of limit order markets may be found e.g. in the introduction of the paper by Luckock (2003).

In the present work, we model the behavior of a risk-averse agent trading in a limit order market maximizing the utility from his terminal wealth. We show that it is not rational to place limit orders given that the agent trades continuously (on-line) and we present a numerical study indicating that limit orders may be rational but the benefit from their usage is negligible given that the agent trades at discrete time instants.

The decision problem

Assume that the agent, equipped with an initial wealth $w_0>0$, chooses between leaving money on a money account and an investment into a risky commodity whose price evolves according to a semimartingale $S(t)$ which is right continuous with left limits (r.c.l.l.). For simplicity, we assume that the interest rate of the money account is zero (more general treatment with a slightly different setting may be found in Šmíd (2007)). We assume the market to be perfectly liquid, i.e. any buy (sell) market order put at any time t results in buying (selling) all the desired amount of the commodity for the price $S(t)$. Finally, we suppose that the agent maximizes his expected utility, measured by a non-decreasing utility function $u,$ at a random horizon $T,$ i.e. he solves

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1 In the present work, however, we assume for simplicity that an arbitrary amount of the commodity may be traded for the best price, see the sequel.

$$\sup \mathrm{E}u(w^x(T)) \quad x \in X, \quad w^x \geq 0, \tag{L}$$

where X – subset of the space of the strategy space Ξ (see Appendix for a definition),
w^x – wealth process given strategy x (see Appendix for a definition),
T – finite stopping time (with respect to the augmented standard filtration (F_t)).

Continuous Time Trading

In the present section, we assume $X=\Xi$, i.e. we allow the agent, at any time, to put any limit order configuration (i.e. any set of buy and/or sell market and/or limit orders).

Even without the detailed definitions (provided in the Appendix) a quite trivial fact may be seen: once the agent is allowed to trade at any (optional) time, he needs not to place any limit orders; in particular, each strategy containing limit orders is (sometimes strictly) dominated by a strategy without limit orders. The reason is simple: putting a limit order is equivalent to putting a market order at the time the limit price is hit by S. Moreover, when the price process exhibits jumps, the price may "jump over" limit orders so that the market order is even more profitable.

We illustrate this fact on a simple example: suppose that $T = \infty$, consider a strategy x consisting of a single buy limit order with unit size and with limit price L, submitted at the time 0. Clearly, the trade, i.e. buying for the price L, takes place if and only if

$$\tau_L < \infty, \qquad \tau_L = \inf\{t : S(t) \leq L\}.$$

Now consider the buy market order with unit size, submitted at time τ_L. Again, the trade is made if and only if $\tau_L < \infty$ but, unlike the previous case, the buying price is $S(\tau_L)$, which is less or equal to L. Hence, strategy[2] y consisting solely of the considered market order is no worse than x, possibly strictly dominating x with a non-zero probability. Our trivial case may be generalized as follows:

Theorem 1. *For any strategy x there exists a strategy y containing no limit orders such that*

$$w^x(t) \leq w^y(t), \quad t \geq 0,$$

where w^z is the wealth process given strategy z (see Appendix for the definition).[3]

Proof. See Šmíd (2007), Proposition 1, (iii).

Together with the monotonicity of the utility function u, Theorem 1 implies that, for each strategy x, there exists a strategy y containing no limit orders such that

$$\mathrm{E}u(w^x(T)) \leq \mathrm{E}u(w^y(T)),$$

i.e. limit orders are useless or even disadvantageous given the setting of the present paragraph.

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[2] Object y is truly a strategy according to our definition thanks to the fact that τ_L is an optional time, see Appendix and Kallenberg (2002), Theorem 7.7.

[3] An assertion analogous to Theorem 1 holds also in the case of a non-zero bid-ask spread (i.e. a different buying and selling price), see Šmíd (2007).

Another consequence of Theorem 1 is that, given the on-line trading, the solution of complicated problem (L) reduces to the "traditional" portfolio selection problem (see e.g. Karatzas and Shreve (1998)) whose solution is known in some cases. For more details, see Šmíd (2007).

Discrete Time Trading

Since on-line trading costs money and time, not all investors trade this way. In the present section, we shall examine the case in which the investor comes to the market only time-to-time; in particular, that he trades at times 0, 1, …, N-1 and that $T=N$, i.e. $X = \{0,1,...,N-1\} \times \xi$ for some set ξ of limit order configurations.

A question arises, whether it is also possible, similarly to the previous section, to reduce the problem (L) with discrete time to the ``traditional'' multi-period portfolio selection problem. Unfortunately, the answer is *no*. We will demonstrate this fact in the following numerical mini-study:

Consider a power utility function, i.e. $u(x) = x^\gamma / w_0^\gamma$ for some $0<\gamma<1$, and suppose S to be the geometrical Brownian motion with unit volatility and the drift equal to –0.3, i.e.

$$S(0) = 1, \quad dS(t)/S(t) = -0.3dt + dW(t),$$

where W is the standard Wiener process. Put N=1, w_0=2 and let ξ be the set of limit order configurations containing no more than two unit buy (limit or market) orders and no sell orders. In this setting, the only two decision variables are the limit prices of the orders - denote them b_1 and b_2 and assume $b_1 \geq b_2$. Since S starts from 1 and since avoiding to put a buy order is equivalent to putting the limit order with zero limit price, it suffices to take $b_1, b_2 \in [0,1]$; problem (L) then may be rewritten as

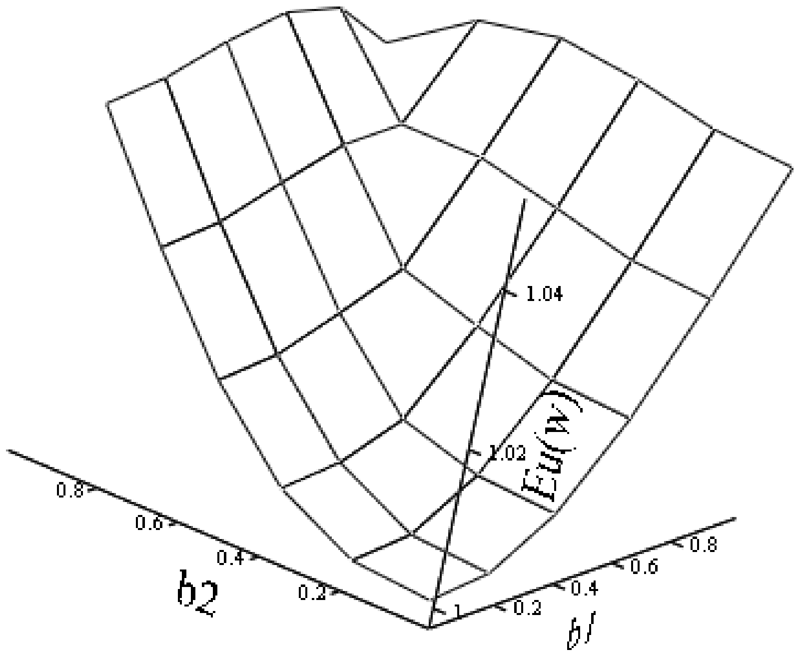
$$\sup_{b_1,b_2 \in [0,1]} \mathrm{E}u\left(w^{b_1,b_2}\right) \tag{L'}$$

where

$$w^{b_1,b_2} = \begin{cases} 2S(1) - b_1 - b_2 + 2 & \text{if both } b_1 \text{ and } b_2 \text{ were hit by } S, \\ S(1) - b_1 + 2 & \text{if only } b_1 \text{ was hit by } S, \\ 2 & \text{otherwise.} \end{cases}$$

By a numerical solution of the problem for different values of parameter γ we may find that, for $\gamma = 0.72$, it is optimal to put one market order and one limit order with the limit price roughly equal to 0.8 (see Fig. 1). Hence, limit orders may outperform market ones given the discrete trading.

Figure 1: Dependence of the expected utility on limit prices for γ=0.72



Source: authorial calculation

Another question is: how much limit orders help? The following table shows the improvement (in percents) of the optimal value of (L') compared to the situation in which only market orders are allowed:

Table 1: Improvement due to limit orders

γ	0.00	0.02	...	0.66	0.68	0.70	0.72	0.74	0.76	…	0.98	1.00
%	0.00	0.00	…	0.00	0.20	0.30	0.50	0.20	0.00	…	0.00	0.00

Source: authorial computation

From the table, it is clear that limit orders are useless for most of the values of the risk-aversion coefficient and that they do not even help much in those cases when the parameters belong to the small interval in which they outperform market orders.[4]

Conclusion

The previous text indicates that to put limit order is mostly irrational; however, this finding contradicts reality (markets are full of limit orders). How can this be explained?

The most straightforward answer is the existence of market makers, who are *paid* for quoting prices which is nothing else but putting limit orders (by the way, their

[4] The results would be similar if we changed drift and volatility of the price process in addition to γ.

existence supports our findings: the fact that organizers of markets *have to* pay money for the presence of limit orders could not be a reaction to nothing else but a lack of them).

However, the presence of market makers does not explain the existence of limit orders fully – there are markets (like KOBOS, a segment of Prague stock exchange) where there are no paid market makers and where limit orders are used widely.

Other explanations of the phenomenon of limit orders could be irrationality of agents, impossibility of immediate reaction or the fact that older orders are usually preferred when fulfilling market orders. However, we are not able to prove any of these hypotheses.

Hence, to find where limit orders come from remains open question.

References

[1] KALLENBERG, O. (2002). *Foundations of Modern Probability*. New York : Springer, 2002.

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[3] LUCKOCK, H. (2003). A steady-state model of the continuous double auction. *Quantitative Finance*, 2003, 3, pp. 385–404.

[4] ŠMÍD, M. (2007). *Dynamic decision problem at limit order market* (2007). [on-line], New York, SSRN, c2007, [cit. 31.3.2007], <http://ssrn.com/abstract=972159>.

Appendix – Rigorous Definitions

Strategy Space

We define the strategy space as

$$\Xi = \{x : x = (\tau_i, \eta_i)_{i \in \mathrm{N}}\}$$

where

- τ_i is an (optional) time of the i-th agent's action;
- $\eta_i = (\beta_{i,j}, \sigma_{i,j}, \alpha_{i,j}, \rho_{i,j})_{j \in \mathrm{N}}$ is the limit order configuration which has been put at time τ_i;
  - $\beta_{i,j} \in [0, \infty)$ is the limit price of the j-th buy order out of the i-th configuration (putting $\beta_{i,j} = A(\tau_i)$ stands for a market order);
  - $\sigma_{i,j} \geq 0$ is the size of the j-th buy order of the i-th configuration;
  - $\alpha_{i,j} \in (0, \infty]$ is the limit price of the j-th sell order of the i-th configuration ($\alpha_{i,j} = B(\tau_i)$ stands for a market order);
  - $\rho_{i,j} \geq 0$ is the size of the j-th sell order of the i-th configuration.

We assume that

- $0 = \tau_1 < \tau_2 < \ldots < \tau_{I_\xi} < \tau_{I_\xi + 1} = \ldots = \infty$ for some $I_\xi \in \mathrm{N}$;
- η_i is measurable with respect to the filtration $F_{\tau i}$ generated by optional time τ_i;
- the limit prices from the same configuration differ for each $i \in \mathrm{N}$.

Wealth Process

Without loss of generality, we may suppose that the agent cancels all the unfulfilled orders at the time of each action. We define the wealth process w^x associated with strategy $x = (\tau_i, (\beta_{i,j}, \sigma_{i,j}, \alpha_{i,j}, \rho_{i,j})_{j \in \mathrm{N}})_{i \in \mathrm{N}} \in \Xi$ as

$$w^x(t) = c(t) + S(t)q(t), \quad t \geq 0.$$

Here, both the cash process c and the commodity holding process q are piece-wise constant right continuous not jumping outside the set $\Theta = \{\vartheta_{i,j} : i, j \in \mathrm{N}\} \cup \{\theta_{i,j} : i, j \in \mathrm{N}\}$ where $\vartheta_{i,j}$ ($\theta_{i,j}$) denotes the hitting time of the limit price of the *j*-th buy (sell) limit order put at the time τ_i, i.e.

$$\vartheta_{i,j} = \inf\{\tau_i \leq t < \tau_{i+1} : S(t) \leq \beta_{i,j}\}, \quad \theta_{i,j} = \inf\{\tau_i \leq t < \tau_{i+1} : B(t) \geq \alpha_{i,j}\}, \quad i, j \in \mathrm{N},$$

and where the jumps of *c* and *q* are defined by

$$\Delta q(\vartheta_{i,j}) = \sigma_{i,j}, \quad \Delta q(\theta_{i,j}) = -\rho_{i,j}, \quad \Delta c(\theta_{i,j}) = -\sigma_{i,j}\alpha_{i,j}, \quad \Delta c(\theta_{i,j}) = \rho_{i,j}\beta_{i,j},$$
$$i, j \in \mathrm{N},$$

($\Delta x(t)$ denotes the jump of *x* at *t,* and it is understood that whenever two ore more hitting tines coincide, the actual jump of *x* is equal to the sum of the corresponding jump values).

Je racionální používat limitní objednávky?

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Abstrakt

Práce je věnována otázce, zda je racionální obchodovat pomocí limitních objednávek. Zjišťujeme, že pokud agent obchoduje on-line, použití limitních objednávek nijak nezlepší jeho optimální očekávaný užitek a může být i nevýhodné. Dále prezentujeme numerickou studii, která naznačuje, že v případě, kdy agent obchoduje pouze v diskrétních okamžicích, sice může být použití limitních objednávek optimální, jejich použití však výsledný očekávaný užitek zvyšuje zanedbatelně.

Klíčová slova: mikrostruktura trhu; trh s limitními objednávkami; výběr portfolia.

Are Limit Orders Rational?

Abstract

We examine whether it is rational to put limit orders in a limit order market. We find that limit orders are not needed and may be even disadvantageous given that the agent trades on-line. Further, we present a numerical study indicating that putting limit orders may be optimal given that the agent trades at discrete times but the benefit from using them in comparison with immediate buying and selling is negligible.

Key words: market microstructure; limit order market; portfolio selection.

JEL classification: C51, G10.